

A Simple Yao-Yao-Based Spanner of Bounded Degree *

Mirela Damian[†]

Abstract

It is a standing open question to decide whether the Yao-Yao structure for unit disk graphs (UDGs) is a length spanner or not. This question is highly relevant to the topology control problem for wireless ad hoc networks. In this paper we make progress towards resolving this question by showing that the Yao-Yao structure is a length spanner for UDGs of bounded aspect ratio. We also propose a new local algorithm, called Yao-Sparse-Sink, based on the Yao-Sink method introduced by Li, Wan, Wang and Frieder, that computes a $(1 + \varepsilon)$ -spanner of bounded degree for a given UDG and for given $\varepsilon > 0$. The Yao-Sparse-Sink method enables an efficient local computation of sparse sink trees. Finally, we show that all these structures for UDGs – Yao, Yao-Yao, Yao-Sink and Yao-Sparse-Sink – have arbitrarily large weight.

1 Introduction

Let $G = (V, E)$ be a connected graph with n vertices embedded in the Euclidean plane. For any pair of vertices $u, v \in V$, an uv -path is defined by a sequence of edges $uu_1, u_1u_2, \dots, u_sv$. A subgraph H of G is a *length spanner* of G if, for all pairs of vertices $u, v \in V$, the length of a shortest uv -path in H is no longer than a constant times the length of a shortest uv -path in G ; if the constant value is t , H is called a *length t -spanner* and t is called *length stretch factor*.

The *power* needed to support a wireless link uv is $|uv|^\beta$, where β is a path loss gradient (a real constant between 2 and 5) that depends on the transmission environment. A subgraph H of a graph G has *power stretch factor* equal to ρ if, for all pairs of vertices u, v in G , the power of a minimum power uv -path in H is no higher than ρ times the power of a minimum power uv -path in G . Li et al. [3] showed that a graph with length stretch factor δ has power stretch factor δ^β , but the reverse is not necessarily true:

Fact 1 [3]. Any subgraph $H \subseteq G$ with length stretch factor δ has power stretch factor δ^β .

The problem of constructing a sparse spanner of a given graph has received considerable attention from researchers in computational geometry and ad-hoc wireless networks; we refer the reader to the recent book by Narasimhan and Smid [6]. The simplest model of a wireless network graph is the Unit Disk Graph (UDG): an edge exists in the graph if and only if the Euclidean distance between its endpoints is no greater than 1.

It is a standing open question to decide whether the Yao-Yao structure for UDGs introduced by Li et al. [3] is a spanner or not. The Yao-Yao graph (also known as *Yao plus reverse Yao*) is based on the Yao graph [8], from which a number of edges are eliminated through a reverse Yao process, to ensure bounded degree at each node. Progress towards resolving this question has been made by Wang and Li [7], who showed that the Yao-Yao graph has constant *power stretch factor* in a *civilized* UDG. For constant $\lambda > 0$, a λ -civilized graph is a graph in which no two nodes are at distance smaller than λ . Most often wireless devices in a wireless network can not be too close, so it is reasonable to model a wireless ad hoc network as a civilized UDG.

*This work has been supported by NSF grant CCF-0728909.

[†]Department of Computer Science, Villanova University, Villanova, PA 19085. E-mail: mirela.damian@villanova.edu.

In this paper we show that the Yao-Yao graph for a *civilized* UDG has constant *length stretch factor* as well. Although several papers refer to a similar result as appearing in [3], to the best of our knowledge there is no version of [3] that publishes this result. We also analyze the bounded degree spanner generated by the *Yao-Sink* technique introduced in [4]. The sink technique replaces each directed star in the Yao graph consisting of all links directed into a node u , by a tree $T(u)$ with sink u of bounded degree. We propose an enhanced technique called *Yao-Sparse-Sink* that filters out some of the edges in the Yao graph prior to applying the sink technique. This enables an efficient local computation of sparse sink trees, more appropriate for highly dynamic wireless network nodes. Our analysis of the Yao-Sparse-Sink method provides additional insight into the properties of the Yao-Yao structure. We also show that all these structures for UDGs – Yao, Yao-Yao, Yao-Sink and Yao-Sparse-Sink – have arbitrarily large weight.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and definitions and discuss previous related work. In Section 3 we show that the Yao-Yao graph is a spanner for UDGs of bounded aspect ratio (in particular, for λ -civilized UDGs). In Section 4 we discuss the Yao-Sink method and, based on this, we propose a new technique called Yao-Sparse-Sink that computes sparse sink trees efficiently. Finally, in Section 5 we show that all these structures for UDGs have unbounded weight.

2 Preliminaries

2.1 Definitions and Notation

Throughout the paper we use the following notation: uv denotes the edge with endpoints u and v ; \vec{uv} denotes the edge directed from *source* node u to *sink* node v ; $|uv|$ denotes the Euclidean distance between u and v ; $p(u \rightsquigarrow v)$ denotes a simple uv -path; and \oplus denotes the concatenation operator. For any nodes u and v , let K_u denote an arbitrary cone with apex u , and $K_u(v)$ denote the cone with apex u containing v . For any edge set E and any cone K_u , let $E \cap K_u$ denote the subset of edges in E incident to u that lie in K_u . Similarly, for a graph G and a cone K_u , $G \cap K_u$ is the subset of edges in G incident to u that lie in K_u . The *aspect ratio* of an edge set E is the ratio of the length of a longest edge in E to the length of a shortest edge in E . The aspect ratio of a graph is defined as the aspect ratio of its edge set.

We assume that each node u has a unique identifier $\text{ID}(u)$. Define the identifier $\text{ID}(\vec{uv})$ of a directed edge \vec{uv} to be the triplet $(|uv|, \text{ID}(u), \text{ID}(v))$. For any pair of directed edges \vec{uv} and $\vec{u'v'}$, we say that $\text{ID}(\vec{uv}) < \text{ID}(\vec{u'v'})$ if and only if one of the following conditions holds:

- (a) $|uv| < |u'v'|$
- (b) $|uv| = |u'v'|$ and $\text{ID}(u) < \text{ID}(u')$
- (b) $|uv| = |u'v'|$ and $\text{ID}(u) = \text{ID}(u')$ and $\text{ID}(v) < \text{ID}(v')$

For an undirected edge uv , define $\text{ID}(uv) = \min\{\text{ID}(\vec{uv}), \text{ID}(\vec{vu})\}$. Note that according to this definition, each edge has a unique identifier. This enables us to order any edge set by increasing ID of edges.

2.2 Previous Work

Yao [8] defined the Yao graph $Y_k(G)$ as follows. At each node $u \in V$, any k equal-separated rays originated at u define k cones; in each cone, pick the edge uv of smallest ID, if such an edge exists, and add to the Yao graph the directed edge \vec{uv} . We call this the YAO-STEP, described in Table 1.

YAO-STEP($G = (V, E), k$)
Set $E_Y \leftarrow \phi$ and $Y_k \leftarrow (V, E_Y)$. For each node u Partition the space into k equal-size cones with apex u of angle $\theta = 2\pi/k$ (assume that each cone is half-open and half-closed). For each node u and each cone K_u such that $E \cap K_u$ is nonempty Pick the edge $uv \in E \cap K_u$ with lowest $\text{ID}(uv)$. Add the directed edge \vec{uv} to E_Y . Output $Y_k = (V, E_Y)$.

Table 1: The Yao step.

It has been shown that the output graph Y_k has maximum node degree $n - 1$ and length stretch factor $\frac{1}{1 - 2 \sin \pi/k}$. The first property (high degree) is the main drawback of the Yao graph. In wireless networks for example, high degree is undesirable because nodes communicating with too many nodes directly may experience large overhead that could otherwise be distributed among several nodes. The Yao-Yao graph YY_k has been proposed in [3] to overcome this shortcoming: at each node u in the Yao graph, discard all directed edges \vec{vu} from each cone centered at u , except for the one with minimum ID. This filtering step is described in Table 2.

REVERSE YAO-STEP($Y_k = (V, E_Y), k$)
Set $E_{YY} \leftarrow E_Y$ and $YY_k \leftarrow (V, E_{YY})$. Use the same cone partition as in the YAO-STEP. For each node v and each cone K_v Eliminate from E_{YY} all edges \vec{uv} with sink v that lie in K_v , except for the one with the smallest ID. Output $YY_k = (V, E_{YY})$ (viewed as an undirected graph).

Table 2: The reverse Yao step.

The output graph YY_k has maximum node degree $2k$, a constant. However, the tradeoff is unclear in that the question of whether YY_k is a spanner or not remains open.

3 YY-Spanner for Civilized UDGs

Note that any UDG of constant aspect ratio Δ is a $1/\Delta$ -civilized UDG, and any λ -civilized UDG has aspect ratio $1/\lambda$. Therefore, from here on will refer to λ -civilized UDGs only. The YY-SPANNER algorithm applied on a UDG $G = (V, E)$ comprises the Yao and reverse-Yao steps:

1. Execute YAO-STEP(G, k). The result is the Yao spanner $Y_k = (V, E_Y)$.
2. Execute REVERSE YAO-STEP(Y_k, k). The result is the Yao-Yao graph $YY_k = (V, E_{YY})$.

We now show that YY_k is a length spanner for any civilized UDG. In proving this, we will make use of the following lemma:

Lemma 1 (Czumaj and Zhao [1]) Let $0 < \theta < \frac{\pi}{4}$ and $t \geq \frac{1}{\cos \theta - \sin \theta}$. Let u, v, z be three points in the plane with $\widehat{vuz} \leq \theta$. Suppose further that $|uz| \leq |uv|$. Then the edge $\{u, z\}$ followed by a t -spanner path from z to v is a t -spanner path from u to v (see Figure 1).

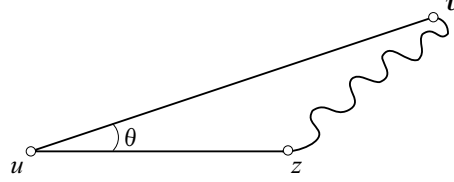


Figure 1: If $\theta < \pi/4$ and $p(z \rightsquigarrow v)$ is a t -spanner path, then $uz \oplus p(z \rightsquigarrow v)$ is a t -spanner path.

Theorem 2 Let $G = (V, E)$ be a λ -civilized graph, and let YY_k be the Yao-Yao structure for G . Then YY_k is a spanner with length stretch factor $t \geq \frac{\lambda}{(\lambda+1)(\cos 2\pi/k - \sin 2\pi/k) - 1}$, for any integer $k > 8$ satisfying the condition $(\cos 2\pi/k - \sin 2\pi/k) > \frac{1}{\lambda+1}$.

Proof: The proof is by induction on the rank of edges in the edge set E ordered by increasing ID. The base case corresponds to the edge $uv \in E$ of rank 0 (i.e., with smallest $\text{ID}(uv)$). Assume without loss of generality that $\text{ID}(uv) = \text{ID}(\vec{uv})$. Since \vec{uv} has the smallest ID among all edges in $K_u(v)$, \vec{uv} gets added to E_Y in the YAO-STEP. Furthermore, since \vec{uv} has the smallest ID among all edges in $E_{YY} \cap K_v(u)$ directed into u , \vec{uv} does not get discarded in the REVERSE YAO-STEP. Thus uv is an edge in YY_k and so the theorem holds for the base case.

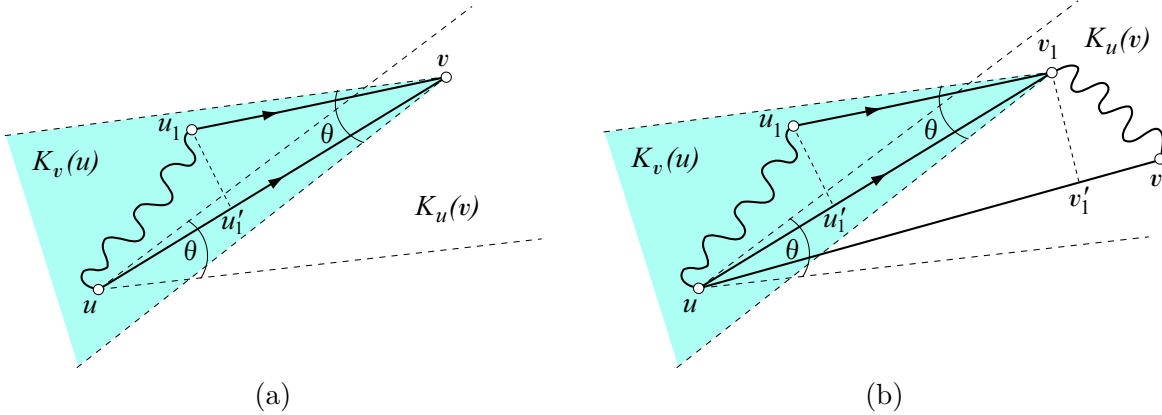


Figure 2: Proof of Theorem 2: (a) Case 1: $p(u \rightsquigarrow v) \leftarrow p(u \rightsquigarrow u_1) \oplus u_1v$; (b) Case 2: $p(u \rightsquigarrow v) \leftarrow p(u \rightsquigarrow u_1) \oplus u_1v_1 \oplus p(v_1 \rightsquigarrow v)$.

The inductive hypothesis tells that YY_k contains t -spanner paths between the endpoints of any edge $uv \in E$ of rank no greater than some value $j \geq 0$. To prove the inductive step, consider the edge $uv \in E$ of rank $j + 1$. Assume without loss of generality that $\text{ID}(uv) = \text{ID}(\vec{uv})$. We discuss two cases, depending on whether \vec{uv} belongs to E_Y or not. Let $\theta = 2\pi/k < \pi/4$.

Case 1: $\vec{uv} \in E_Y$. If $\vec{uv} \in E_{YY}$ the proof is finished, so assume the opposite. Note that $\vec{uv} \notin E_{YY}$ happens when v eliminates \vec{uv} in the REVERSE YAO-STEP in favor of another edge $\vec{u_1v}$, with $\text{ID}(\vec{u_1v}) < \text{ID}(\vec{uv})$ (see Figure 2a). Since u and u_1 both lie in a same cone K_v , we have

that $\widehat{uvu_1} \leq \theta < \pi/4$ and therefore $|u_1u| < |uv|$. It follows that $\text{ID}(u_1u) < \text{ID}(uv)$. Conform the inductive hypothesis, YY_k contains a t -spanner path $p(u \rightsquigarrow u_1)$ from u to u_1 . These together with Lemma 1 show that $p(u \rightsquigarrow u_1) \oplus u_1v$ is a t -spanner path from u to v in YY_k .

Case 2: $\overrightarrow{uv} \notin E_Y$. Let $uv_1 \in E \cap K_u(v)$ be the edge selected by u in the YAO-STEP. Thus we have that $\text{ID}(uv_1) < \text{ID}(uv)$ and therefore $|uv_1| \leq |uv|$. If $\overrightarrow{uv_1} \in E_{YY}$, then arguments similar to the ones used for Case 1 show that $uv_1 \oplus p(v_1 \rightsquigarrow v)$ is a t -spanner path from u to v in YY_k ; the existence of a t -spanner path $p(v_1 \rightsquigarrow v)$ in YY_k is ensured by the inductive hypothesis.

Consider now the case where $\overrightarrow{uv_1} \notin E_{YY}$. Since $\overrightarrow{uv_1} \in E_Y$ and $\overrightarrow{uv_1} \notin E_{YY}$, the edge $\overrightarrow{uv_1}$ must have been eliminated by v_1 in the REVERSE YAO-STEP in favor of another edge $\overrightarrow{u_1v_1}$, with $\text{ID}(u_1v_1) < \text{ID}(uv_1)$ (refer to Figure 2b). Let u'_1 be the projection of u_1 on uv_1 . By the triangle inequality,

$$|uu_1| \leq |uu'_1| + |u'_1u_1| = |uv_1| - |u'_1v_1| + |u'_1u_1| \leq |uv_1| - |u_1v_1| \cos \theta + |u_1v_1| \sin \theta. \quad (1)$$

Similarly, if v'_1 is the projection of v_1 on uv , we have

$$|v_1v| \leq |vv'_1| + |v'_1v_1| = |uv| - |uv'_1| + |v'_1v_1| \leq |uv| - |uv_1| \cos \theta + |uv_1| \sin \theta. \quad (2)$$

Since $|uu_1| < |uv_1| \leq |uv|$ and $|v_1v| < |uv|$, YY_k contains t -spanner paths $p(u \rightsquigarrow u_1)$ and $p(v_1 \rightsquigarrow v)$ (by the inductive hypothesis). Let $P_1 = p(u \rightsquigarrow u_1) \oplus p(v_1 \rightsquigarrow v)$. We show that the path $P = P_1 \oplus u_1v_1$ is a t -spanner path from u to v , thus proving the inductive step. The length of P_1 is

$$|P_1| \leq t(|uu_1| + |v_1v|).$$

Substituting inequalities (1) and (2) yields

$$|P_1| \leq t|uv| + t|uv_1|(1 - \cos \theta + \sin \theta) - t|u_1v_1|(\cos \theta - \sin \theta). \quad (3)$$

Thus the length of $P = P_1 \oplus u_1v_1$ is

$$|P| \leq t|uv| + t|uv_1|(1 - \cos \theta + \sin \theta) - |u_1v_1|(t \cos \theta - t \sin \theta - 1). \quad (4)$$

Since the input graph G is λ -civilized, we have that $|u_1v_1| \geq \lambda$. This along with the inequality (4) and the fact that $|uv_1| \leq 1$ implies

$$|P| \leq t|uv| + (t(1 - \cos \theta + \sin \theta) - \lambda(t \cos \theta - t \sin \theta - 1)).$$

Note that the second term on the right side of the inequality above is non-positive for any $t \geq \frac{\lambda}{(\lambda+1)(\cos \theta - \sin \theta) - 1}$ and for any θ satisfying the condition $\cos \theta - \sin \theta > \frac{1}{\lambda+1}$. This completes the proof. \square

Theorem 2 implies that, for fixed small $\lambda > 0$ and for any $\varepsilon > 0$, one can choose θ such that $\cos \theta - \sin \theta = \frac{\lambda + \varepsilon + 1}{(\lambda+1)(\varepsilon+1)} > \frac{1}{\lambda+1}$, to produce a t -spanner YY_k with $t \geq \frac{\lambda}{(\lambda+1)(\cos \theta - \sin \theta) - 1} = 1 + \varepsilon$. So we have the following result:

Corollary 3 *The YY_k structure produced by the YY-SPANNER algorithm for a given civilized UDG is a spanner with maximum degree $2k$, length stretch factor $(1+\varepsilon)$, and power stretch factor $(1+\varepsilon)^\beta$, for any real $\varepsilon > 0$ and integer $k > 8$ satisfying the condition $\cos 2\pi/k - \sin 2\pi/k = \frac{\lambda + \varepsilon + 1}{(\lambda+1)(\varepsilon+1)}$.*

4 Efficient Local YAO-SPARSE-SINK Algorithm for UDGs

We have established in Section 3 that the Yao-Yao graph is a length spanner for civilized UDGs. The question of whether the Yao-Yao graph is a length spanner for arbitrary UDGs remains open. In order to guarantee both bounded degree and the length spanner property, Li et al. [4] suggest a sparse topology, called *Yao-Sink*. Let $G = (V, E)$ be a UDG. The Yao-Sink algorithm applied on G consists of two steps: (1) Execute YAO-STEP(G, k) to produce the Yao spanner $Y_k = (V, E_k)$, and (2) Execute SINK-STEP(Y_k, k) to reduce the degree of Y_k . The SINK-STEP is described in detail in Table 3. In [5] the authors show that the output YS_k generated by the Yao-Sink method has maximum degree $k(k+2)$ and length stretch factor $\left(\frac{1}{1-2\sin\pi/k}\right)^2$. In fact, the authors show a more general result that applies to *mutual inclusion graphs*, which allow for non-uniform transmission ranges at nodes.

SINK-STEP($Y_k = (V, E_Y), k$)
<p>Use the same cone partition as in the YAO-STEP.</p> <ol style="list-style-type: none"> 1. Set $E_{YS} \leftarrow \emptyset$ and $YS_k \leftarrow (V, E_{YS})$. 2. For each node v and each cone K_v <p>{Build the tree $T(v)$ corresponding to K_v.}</p> <ol style="list-style-type: none"> 2.1 Let I be the set of vertices u such that $\vec{uv} \in E_Y \cap K_v$. Set $I(v) \leftarrow I$. Initialize the ordered vertex sequence $J \leftarrow (v)$. 2.2 Initialize $T(v) \leftarrow \emptyset$. Repeat until I is empty <ol style="list-style-type: none"> 2.2.1 Remove the first vertex u from the sequence J. 2.2.2 For each cone K_u <p>Let $w \in I(u) \cap K_u$ be the node that minimizes $ID(\vec{wu})$ (if any). Add \vec{wu} to $T(v)$ and move w from I to J. Set $I(w) \leftarrow I(u) \cap K_u$.</p> 2.3 Add all edges of $T(v)$ to E_{YS}. <p>Output $YS_k = (V, E_{YS})$ (viewed as an undirected graph).</p>

Table 3: The Sink step.

The following two lemmas (Lemmas 4 and 5) identify two important properties of the output spanner YS_k generated by the Yao-Sink method. Specifically, they show the existence of a particular path in YS_k corresponding to each Yao edge removed in the SINK-STEP.

Lemma 4 *For each edge $\vec{uv} \in E_Y$, there is a uv -path $\Pi = w_0w_1, w_1w_2, \dots, w_{h-1}w_h$ in $K_v(u)$, with $w_0 = v$ and $w_h = u$, such that $\vec{w_iw_{i-1}} \in E_{YS} \cap K_{w_{i-1}}(u)$ and $ID(\vec{w_iw_{i-1}}) < ID(\vec{uw_{i-1}})$, for each $i = 1, 2, \dots, h$.*

Proof: Let $I = I(v)$ be the vertex set defined in Step 2.1 of SINK-STEP for node v and cone $K_v(u)$. If $u \in I(v) \cap K_v(u)$ minimizes $ID(\vec{uv})$, then the path sought is $\Pi = vu$ and the proof is finished. Otherwise, let $\Pi = w_0w_1, w_1w_2, \dots, w_{p-1}w_p$ be a longest path in $K_v(u)$ that satisfies the conditions of the lemma: $\vec{w_iw_{i-1}} \in E_{YS} \cap K_{w_{i-1}}(u)$ and $ID(\vec{w_iw_{i-1}}) < ID(\vec{uw_{i-1}})$, for each $i = 1, 2, \dots, p$. We prove by contradiction that $w_p = u$. Assume to the contrary that w_p and u are distinct. Since $\vec{w_pw_{p-1}} \in E_{YS}$, it must be that $w_p \in I(w_{p-1})$. Furthermore, since w_p and u lie in a same cone $K_{w_{p-1}}(u)$, the set $I(w_p)$ defined in Step 2.2.2 of the SINK-STEP is $I(w_p) = I(w_{p-1}) \cap K_{w_{p-1}}(u)$ and includes both w_p and u .

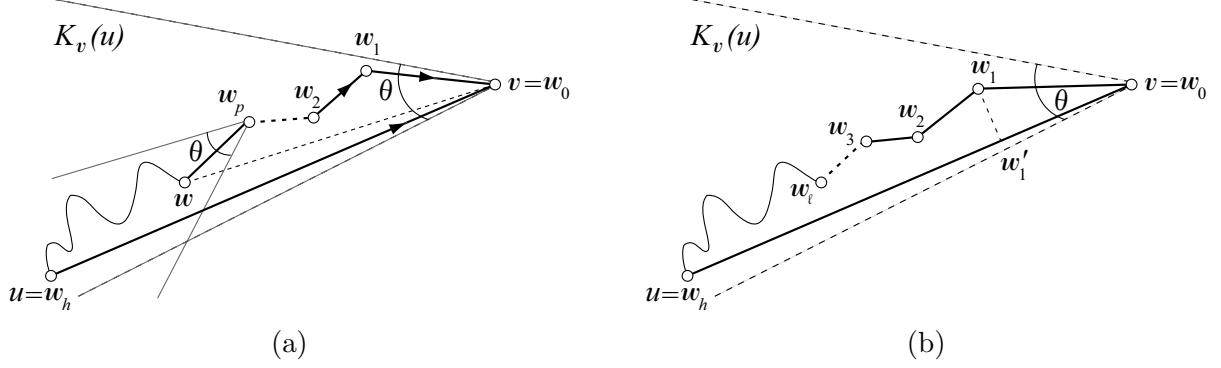


Figure 3: (a) Lemma 4: path $\Pi = w_0w_1, w_1w_2, \dots, w_{h-1}w_h$ (b) Lemma 5: $\widehat{w_1uv} \leq \theta$.

Consider now the instance when w_p gets processed (i.e, it gets removed from J in Step 2.2.1 of SINK-STEP). See Figure 3a. First observe that $I(w_p) \cap K_{w_p}(u)$ is nonempty, since it contains at least the node u . This implies that there exists $w \in I(w_p) \cap K_{w_p}(u)$ that minimizes $\text{ID}(\overrightarrow{uw_p})$. It follows that $\overrightarrow{ww_p} \in E_{YS} \cap K_{w_p}(u)$ and either $w = u$ or $\text{ID}(\overrightarrow{ww_p}) < \text{ID}(\overrightarrow{uw_p})$. Either case contradicts our assumption that Π is a longest path that satisfies the conditions of the lemma. \square

In the context of Lemma 4, we next prove the existence of a long enough subpath of Π from v to one of the vertices $w_\ell \in \Pi$ that closely approximates the direct link vw_ℓ .

Lemma 5 *Let $\Pi = w_0w_1, w_1w_2, \dots, w_{h-1}w_h$, with $w_0 = v$ and $w_h = u$, be the path identified in Lemma 4 corresponding to a given edge $\overrightarrow{uv} \in E_Y$. Then there exists $\ell \leq h$ such that $|vw_\ell| \geq |uv|/(2 \cos \theta)$ and*

$$\sum_{i=0}^{\ell-1} |w_iw_{i+1}| \leq \frac{|vw_\ell|}{\cos 2\theta}.$$

Proof: Let $\ell \leq h$ be the smallest index in the sequence $1, 2, \dots, h$ such that $|vw_\ell| \geq |uv|/(2 \cos \theta)$. Since $|vw_h| = |uv| > |uv|/(2 \cos \theta)$, such an index always exists. Let w'_i be the projection of w_i on uv , for each i . We first prove that the following invariant holds:

- (a) $\widehat{w_i v u} \leq \theta$, for each $i = 0, 1, \dots, \ell$.
- (b) $\widehat{w_i u v} \leq \theta$, for each $i = 1, \dots, \ell - 1$.
- (c) $|w_iw_{i-1}| \leq |w'_iw'_{i-1}|/\cos 2\theta$, for each $i = 1, \dots, \ell$.

Property (a) follows immediately from the fact that w_i and u belong to a same cone $K_v(u)$, for each i . The proof for properties (b) and (c) is by induction on the index i . The base case corresponds to $i = 1$ (i.e, $\Pi = w_0w_1$). See Figure 3b. We prove that $\widehat{w_1uv} \leq \theta$ (claim (b) for the case when $\ell \geq 2$). First observe that $|w_1v| < |uv|/(2 \cos \theta)$, otherwise it would contradict our choice of ℓ . Thus we have that

$$\tan \widehat{w_1uv} = \frac{|w_1w'_1|}{|uv| - |w'_1v|} = \frac{|w_1v| \sin \widehat{w_1vu}}{|uv| - |w_1v| \cos \widehat{w_1vu}} \leq \frac{|w_1v| \sin \theta}{|uv| - |w_1v| \cos \theta} < \tan \theta.$$

It follows that $\widehat{w_1uv} < \theta$, so claim (b) holds. For claim (c), note that $|vw_1| \leq |vw'_1|/\cos \theta < |vw'_1|/\cos 2\theta$. Assume that the claim holds for any index less than i , for some $i > 1$. To prove the inductive step, consider a path $\Pi = w_0w_1, \dots, w_{i-1}w_i$, with $i \leq \ell$. We distinguish three cases:

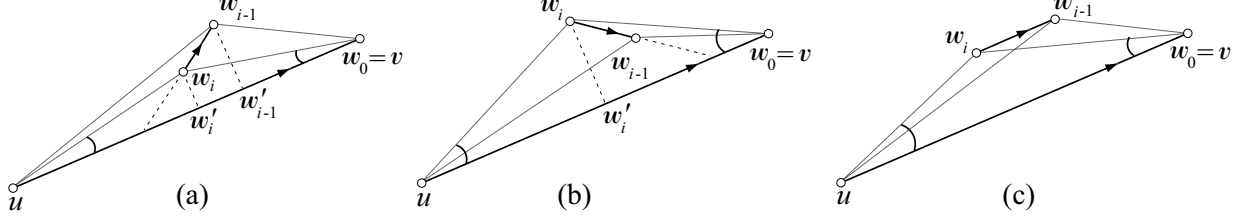


Figure 4: Lemma 5 proof: (a) $w_i \in \triangle uw_{i-1}v$ (b) $w_{i-1} \in \triangle uw_iv$ (c) Neither (a) nor (b) holds.

- (i) $w_i \in \triangle uw_{i-1}v$ (see Figure 4a). Then $\widehat{w_i v u} < \widehat{w_{i-1} v u} \leq \theta$ (this latter inequality is true by the inductive hypothesis). Also note that the angle formed by $w_{i-1}w_i$ and uv is no greater than $\widehat{w_{i-1}w_i u} + \widehat{w_{i-1}u v} \leq 2\theta$ and therefore $|w_i w_{i-1}| \leq |w'_i w'_{i-1}| / \cos 2\theta$.
- (ii) $w_{i-1} \in \triangle uw_iv$ (see Figure 4b). We show that $\widehat{w_i u v} \leq \theta$ (claim (b) for the case when $i \leq \ell - 1$). First observe that the condition $|w_i v| < |uv| / (2 \cos \theta)$ must hold for each $i \leq \ell - 1$; otherwise, we could find a lower index $i < \ell$ satisfying the condition $|v w_i| \geq |uv| / (2 \cos \theta)$, contradicting our choice of ℓ . As before, we have that

$$\tan \widehat{w_i u v} = \frac{|w_i w'_i|}{|uv| - |w'_i v|} = \frac{|w_i v| \sin \widehat{w_i v u}}{|uv| - |w_i v| \cos \widehat{w_i v u}} \leq \frac{|w_i v| \sin \theta}{|uv| - |w_i v| \cos \theta} < \tan \theta.$$

It follows that $\widehat{w_i u v} < \theta$. Also note that in this case the angle formed by $w_{i-1}w_i$ and uv is no greater than $\widehat{w_i w_{i-1} u} < \theta$ and therefore $|w_i w_{i-1}| \leq |w'_i w'_{i-1}| / \cos \theta < |w'_i w'_{i-1}| / \cos 2\theta$.

- (iii) Neither (i) nor (ii) holds (see Figure 4c). Arguments identical to the ones used in case (ii) above show that $\widehat{w_i u v} \leq \theta$. Also note that the angle formed by $w_{i-1}w_i$ and uv is no greater than $\max\{\widehat{w_{i-1}u v}, \widehat{w_i v u}\} \leq \theta$ and therefore $|w_i w_{i-1}| \leq |w'_i w'_{i-1}| / \cos \theta < |w'_i w'_{i-1}| / \cos 2\theta$.

We have shown that $|w_i w_{i-1}| \leq |w'_i w'_{i-1}| / \cos 2\theta$ for each $i = 1, 2, \dots, \ell$. Summing up over i yields

$$\sum_{i=0}^{\ell-1} |w_i w_{i+1}| \leq \frac{|w'_\ell v|}{\cos 2\theta} < \frac{|w_\ell v|}{\cos 2\theta}.$$

This completes the proof. \square

We will show that Lemmas 4 and 5 enable us to discard some Yao edges from Y_k prior to executing the SINK-STEP, without compromising the spanner property. This leads to the construction of efficient sparse sink trees in the SINK-STEP. Table 4 describes our method called YAO-SPARSE-SINK that incorporates this intermediate edge filtering step. In the filtering step, each node u partitions the set of Yao edges incident to u into a number of subsets, such that all edges in a same subset F_i have similar sizes. The aspect ratio of each subset F_i is controlled by the input parameter $r > 1$. From each subset F_i , only the Yao edge with smallest ID is carried on to the SINK-STEP; all other Yao edges from F_i are discarded.

It can be verified that the result YE_k of the filtering step is a spanner for G of maximum degree $O(\log \Delta)$, where Δ is the aspect ratio of G . Because of space constraints we skip this proof and turn instead to showing that the output YES_k generated by the YAO-SPARSE-SINK method is a spanner of constant maximum degree. Intuitively, if YES_k contains short paths between the endpoints of an edge processed in the SINK-STEP, then YES_k contains short paths between the endpoints of all nearby edges of similar sizes.

Algorithm YAO-SPARSE-SINK($G = (V, E), k, r$)	
1.	Execute YAO-STEP(G, k). The result is the Yao spanner $Y_k = (V, E_Y)$.
2.	For each node $v \in V$ and each cone K_v Let $F \subseteq E_Y \cap K_v$ be the subset of Yao edges from K_v directed into v Let $\vec{uv} \in F$ be the edge of minimum ID. Let Δ be the aspect ratio of F . Partition F into disjoint subsets F_1, F_2, \dots, F_s , with $s = \lceil \log_r \Delta \rceil$, such that $F_i = \{ab \in F \mid uv r^{i-1} \leq ab < uv r^i\}.$ For each $i = 1, 2, \dots, s$ Add to E_{YE} (initially \emptyset) the edge from F_i of smallest ID. Result is $YE_k = (V, E_{YE})$ of degree $O(\log \Delta)$.
3.	Execute SINK-STEP(YE_k, k). The result is $YES_k = (V, E_{YES})$.
Output YES_k (viewed as an undirected graph).	

Table 4: The Yao-Sparse-Sink algorithm.

Theorem 6 *Let $G = (V, E)$ be a UDG and let $r > 1$, $k \geq 8$, $\theta = 2\pi/k$ and $\lambda = \frac{1}{2r \cos \theta}$ be constants such that $(\cos \theta - \sin \theta) > \frac{\lambda}{\lambda+1}$. When run with these values of r and k , the output of the YAO-SPARSE-SINK algorithm is a t -spanner of degree $k(k+2)$, for any $t \geq \frac{\lambda / \cos(2\theta)}{(\lambda+1)(\cos \theta - \sin \theta) - 1}$.*

Proof: The degree of YES_k is no greater than the degree of the Yao-Sink spanner, which is $k(k+2)$ [5]. We now prove that YES_k is a t -spanner. The proof is by induction on the rank of edges in the set E ordered by increasing ID. The base case corresponds to the edge $uv \in E$ of minimum ID. Arguments similar to the ones used for the base case in Theorem 2 show that $uv \in YES_k$.

The inductive hypothesis tells that YES_k contains t -spanner paths between the endpoints of any edge $uv \in E$ whose rank is no greater than some value $j \geq 0$. To prove the inductive step, consider the edge $uv \in E$ of rank $j+1$. Assume without loss of generality that $\text{ID}(uv) = \text{ID}(\vec{uv})$. We discuss two cases, depending on whether \vec{uv} belongs to E_Y or not.

Case 1: $\vec{uv} \in E_Y$. Assume first that $\vec{uv} \in E_{YE}$. By Lemma 5, YES_k contains an edge $\vec{w_1v} \in K_v(u)$. This implies that $\widehat{uvw_1} \leq \theta < \pi/4$. Furthermore, since $\text{ID}(w_1v) < \text{ID}(uv)$ (and therefore $|w_1v| \leq |uv|$), we have that $|uw_1| < |uv|$ (see Figure 5a). Thus we can use the inductive hypothesis to show that YES_k contains a t -spanner path $p(u \rightsquigarrow w_1)$. By Lemma 1, $p(u \rightsquigarrow w_1) \oplus w_1v$ is a t -spanner path in YES_k from u to v .

Assume now that $\vec{uv} \notin E_{YE}$. Let F be the edge set corresponding to cone $K_v(u)$ and let i be such that $\vec{uv} \in F_i$. Since $uv \notin E_{YE}$ there is an edge $\vec{u_1v} \in F_i$ of smaller ID that gets added to E_{YE} . Note however that u_1 and u belong to one same cone $K_v(u)$ (see Figure 5b). By the same arguments as above, there is $\vec{w_1v} \in K_v(u)$ corresponding to the edge $\vec{u_1v} \in E_{YE}$ which enables us to identify the t -spanner path $p(u \rightsquigarrow w_1) \oplus w_1v$ from u to v in YES_k .

Case 2: $\vec{uv} \notin E_Y$. Let $uv_1 \in E_Y$ be the edge selected by u in the YAO-STEP. Thus we have that $\text{ID}(uv_1) < \text{ID}(uv)$ and therefore $|uv_1| \leq |uv|$. If $\vec{uv_1} \in E_{YES}$, then arguments similar to the ones used for Case 1 show that $uv_1 \oplus p(v_1 \rightsquigarrow v)$ is a t -spanner path from u to v in YES_k .

Consider now the case when $\vec{uv_1} \notin E_{YES}$. Let F be the edge set corresponding to cone $K_{v_1}(u)$ and let i be such that $\vec{uv_1} \in F_i$. Let $\vec{u_1v_1} \in F_i$ be the edge that gets added to E_{YE} in Step 2

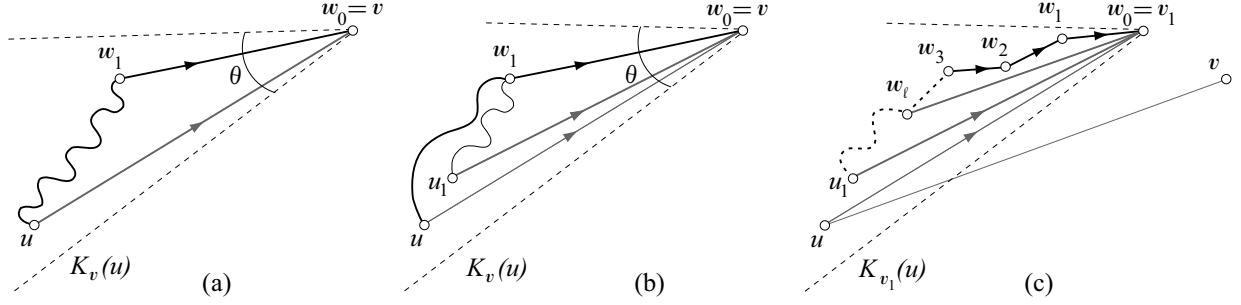


Figure 5: Proof of Theorem 6: (a) Case $\vec{uv} \in E_{YE}$ (b) Case $\vec{uv} \in E_Y \setminus E_{YE}$ (c) Case $\vec{uv} \notin E_Y$.

of the YAO-SPARSE-SINK algorithm. Since both uv_1 and u_1v_1 belong to a same set F_i , and since $\text{ID}(\overrightarrow{u_1v_1}) \leq \text{ID}(\overrightarrow{uv_1})$ (equality happens when $u_1 = u$), we have that

$$|u_1v_1| \geq |uv_1|/r. \quad (5)$$

Lemma 5 indicates that, corresponding to the edge $\overrightarrow{u_1v_1} \in E_{YE}$, there exists a path $P_0 \in YES_k \cap K_{v_1}(u_1)$ extending from $w_0 = v_1$ to some vertex w_ℓ , such that

$$|P_0| \leq |w_\ell v_1|/\cos 2\theta. \quad (6)$$

$$|w_\ell v_1| \geq |u_1v_1|/(2\cos \theta) \quad (7)$$

Now note that, since $|v_1w_\ell| \leq |v_1u_1|$ and since $\widehat{w_\ell v_1 u} \leq \theta \leq \pi/4$, we have that $|uw_\ell| < |uv_1| \leq |uv|$. Similarly, $|v_1v| \leq |uv|$. Thus we can use the inductive hypothesis to claim the existence of t -spanner paths $p(u \rightsquigarrow w_\ell)$ and $p(v_1 \rightsquigarrow v)$. Let $P_1 = p(u \rightsquigarrow w_\ell) \oplus p(v_1 \rightsquigarrow v)$. We show that $P = P_0 \oplus P_1$ is a t -spanner path from u to v . Calculations identical to the ones used to derive the inequality (3) yield

$$|P_1| \leq t|uv| + t|uv_1|(1 - \cos \theta + \sin \theta) - t|w_\ell v_1|(\cos \theta - \sin \theta).$$

This along with (6) shows that the length of $P = P_0 \oplus P_1$ is

$$|P| \leq t|uv| + t|uv_1|(1 - \cos \theta + \sin \theta) - |w_\ell v_1|(t \cos \theta - t \sin \theta - 1/\cos 2\theta).$$

Substituting (5) and (6) yields

$$|P| \leq t|uv| + (t|uv_1|(1 - \cos \theta + \sin \theta) - \frac{|uv_1|}{2r \cos \theta}(t \cos \theta - t \sin \theta - 1/\cos 2\theta)). \quad (8)$$

Note that the second term in the right hand side of the inequality (8) is non-positive for any $t \geq \frac{\lambda/\cos 2\theta}{(\lambda+1)(\cos \theta - \sin \theta) - 1}$ and for any θ satisfying the condition $\cos \theta - \sin \theta > \frac{1}{\lambda+1}$. \square

Arguments similar to the ones used for Corollary 3 show that, for appropriate values of r and k corresponding to a fixed $\varepsilon > 0$, $YY S_k$ is a $(1 + \varepsilon)$ -spanner.

Efficient Local Implementation. For a local implementation of the YAO-STEP, the authors propose in [3] to have each sink node u build $T(u)$ and then broadcast $T(u)$ to all nodes in $T(u)$. It can be easily verified that, for each node u and each cone K_u , the neighbors of u that lie in K_u (including u) form a clique. This suggests a more efficient alternate local implementation of the

YAO-STEP: each node collects the coordinate information from its immediate neighbors, then simulates the execution of the YAO-STEP locally, on the collected neighborhood. This implementation avoids broadcasting messages of size $O(n)$ (encoding the sink trees) by each node, thus saving some battery power. This idea can be extended to the YAO-SPARSE-SINK algorithm as well: each node collects its neighborhood information in one round of communication, then simulates the execution the YAO-SPARSE-SINK algorithm on the collected neighborhood.

5 Total Weight of Y_k , YY_k , YS_k and YES_k

Define the total *weight* $wt(G)$ of a graph G as the sum of the lengths of its constituent edges. We first show that the total weight of the Yao graph Y_k constructed by YAO-STEP is arbitrarily high compared to the weight of the Minimum Spanning Tree (MST) for V . Although this result is fairly straightforward, to the best of our knowledge it has not appeared in the literature.

Theorem 7 *Let G be a UDG and let $Y_k = \text{YAO-STEP}(G, k)$. Then $wt(Y_k) = \Omega(n) \cdot wt(MST)$.*

Proof: Consider a set of $n = 2s$ nodes equally distributed along the top and bottom sides of a unit square, as in Figure 6. Let u_1, u_2, \dots, u_s denote the top nodes and v_1, v_2, \dots, v_s the bottom nodes. Observe that for each node u_i , the angular distance between any of its left/right neighbors and v_i

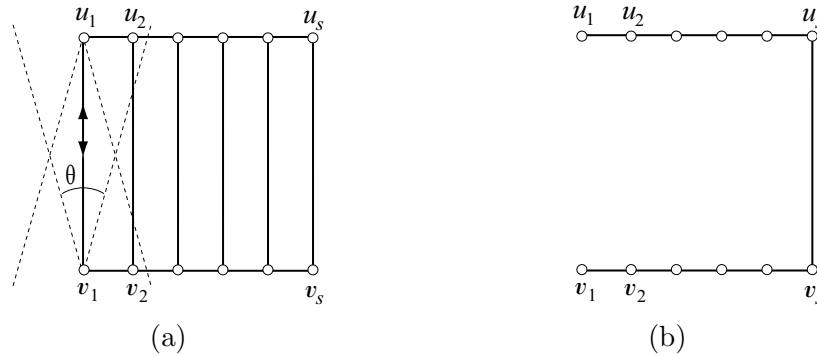


Figure 6: Y_k has unbounded weight: (a) $wt(Y_k) = n/2 + 2$; (b) $wt(MST) = 3$.

is $\pi/2$. This means that the only edge incident to u_i that lies in the cone of angle $2\pi/k \leq \pi/3$ (for $k \geq 6$) centered at u_i and containing v_i is $u_i v_i$ (see Figure 6a). Consequently, u_i adds $\overrightarrow{u_i v_i}$ to E_Y in the YAO-STEP. Similarly, v_i adds $\overrightarrow{v_i u_i}$ to E_Y . Thus the total weight of Y_k is no less than

$$\sum_{i=1}^s |u_i v_i| = n/2.$$

However, the weight of the spanning tree illustrated in Figure 6b is 3. This completes the proof. \square

We have shown that, for any node u in the topology from Figure 6a, at most one edge from Y_k lies in any cone K_u centered at u . This implies that:

- (a) No edges from Y_k get discarded in the REVERSE YAO-STEP.
- (a) No edges from Y_k get discarded in the filtering step (Step 2) of YAO-SPARSE-SINK.

- (b) No edges from Y_k get altered in the SINK-STEP.

This implies that YY_k , YS_k and YES_k are all identical to Y_k and therefore have unbounded weight as well.

It is worth noting that any civilized UDG $G = (V, E)$ has weight within a constant factor of $wt(MST(V))$ and therefore the structures Y_k , YY_k , YS_k and YES_k for civilized UDGs have bounded weight as well. This follows immediately from a result obtained by Das et al. [2]:

Fact 2 (Theorem 1.2 in [2]). If a set of line segments E satisfies the isolation property, then $wt(E) = O(1) \cdot wt(MST)$.

A set of line segments E is said to satisfy the *isolation property* if each segment $uv \in E$ can be associated with a cylinder B of height and width equal to $c|uv|$, for some constant $c > 0$, such that the axis of B is a subsegment of uv and B does not intersect any line segment other than uv . In the case of λ -civilized UDGs, this property is satisfied by $c = \lambda$.

6 Conclusions

We have shown that the Yao-Yao graph is a spanner for UDGs of bounded aspect ratio. We have also proposed an extension of the Yao-Sink method, called Yao-Sparse-Sink, that enables an efficient local computation of sparse sink trees. The Yao-Sparse-Sink method is preferable to the Yao-Sink method for topology control in highly dynamic wireless environments. Our analysis of the Yao-Sparse-Sink method provides additional insight into the properties of the Yao-Yao structure. However, the main question of whether the Yao-Yao graph for arbitrary UDGs is a length spanner or not remains open.

Acknowledgement. We thank Michiel Smid for helpful discussions on these problems.

References

- [1] A. Czumaj and H. Zhao. Fault-tolerant geometric spanners. *Discrete & Computational Geometry*, 32(2):207–230, 2004.
- [2] Gautam Das, Giri Narasimhan, and Jeffrey Salowe. A new way to weigh malnourished Euclidean graphs. In *SODA '95: Proceedings of the sixth annual ACM-SIAM symposium on Discrete algorithms*, pages 215–222, Philadelphia, PA, USA, 1995. Society for Industrial and Applied Mathematics.
- [3] X. Li, P. Wan, Y. Wang, and O. Frieder. Sparse power efficient topology for wireless networks. In *HICSS'02: Proc. of the 35th Annual Hawaii International Conference on System Sciences*, volume 9, page 296.2, 2002.
- [4] X. Y. Li, P. J. Wan, and Y. Wang. Power efficient and sparse spanner for wireless ad hoc networks. In *ICCCN '01: IEEE International Conference on Computer Communications and Networks*, 2001.
- [5] Xiang-Yang Li, Wen-Zhan Song, and Yu Wang. Efficient topology control for ad-hoc wireless networks with non-uniform transmission ranges. *Wireless Networks*, 11(3):255–264, 2005.
- [6] Giri Narasimhan and Michiel Smid. *Geometric Spanner Networks*. Cambridge University Press, 2007.
- [7] Y. Wang and X. Y. Li. Localized construction of bounded degree and planar spanner for wireless ad hoc networks. In *Proc. of the Joint Workshop on Foundations of Mobile Computing*, pages 59–68, 2003.
- [8] A.C.-C. Yao. On constructing minimum spanning trees in k -dimensional spaces and related problems. *SIAM Journal on Computing*, 11(4):721–736, 1982.